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Generalized derivations of Rings in the center

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Abstract— Ashraf, Rehman, Bell, Martindale and Daif have obtained commutativity of prime and semiprime rings with derivations satisfying certain polynomial constants. Ashraf and Nadeem established that a prime ring R with nonzero ideal A must be commutative if it admits a derivation d satisfying either of the properties $d(xy) + xy \in U$ or $d(xy) - xy \in U$ for all $x, y \in R$. Hvala initiated the algebraic study of generalized derivation and extended some results concerning derivation to generalized derivation. In 2007 Ashraf, Asma Ali and Shakir Ali studied commutativity of a prime associative ring in which the generalized derivation F satisfies certain properties. In this paper we prove the commutativity of a prime nonassociative ring R satisfying any one of the following properties :

(i) $F(xy) - xy \in U$, (ii) $F(xy) + xy \in U$,(iii) $F(x)F(y) - xy \in U$ and (iv) $F(x)F(y) + xy \in U$ for all x, y in R, where F is a generalized derivation on R and U is the center of R.

Index Terms— Center, Prime ring, derivation, Generalized derivation.

1 INTRODUCTION

Throughout this paper R denotes a prime nonassociative ring satisfying $[xy, z] = x[y, z] + [x, z]y$ for all x, y, z in R. A ring R is prime if $aRb = (0)$ implies that $a = 0$ or $b = 0$. An additive mapping $d : R \rightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all x, y in R. An additive mapping $F : R\rightarrow R$ is said to be a generalized derivation on R if there exists a derivation $d : R\rightarrow R$ such that $F(xy) = F(x)y + xd(y)$ holds for all x, y in R.

2 RESULTS

Theorem 1 : Let R be a prime nonassociative ring satisfying $[xy, z] = x[y, z] + [x, z]y$ for all x, y, z in R and A be an associative nonzero ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d such that $F(xy) - xy \in U$ for all $x, y \in A$, then R is commutative.

Proof : If $F = 0$, then $xy\in U$ for all x, y in A.In particular $[xy, x] = 0$ for all $x, y\in A$ and hence $x[y, x] = 0$.By replacing y with yz, we get $xy[z, x] = 0$ for all $x, y, z\in A$.
Hence it follows that $xRA[z, x] = (0)$ for all $x, z\in A$.
Thus the primeness of R forces for each $x\in A$, either $x = 0$ or $A[x, z] = (0)$.
But $x = 0$ also implies that $A[x, z] = (0)$.
Hence in both cases we find that $A[x, z] = (0)$ for all $z\in A$, that is,$AR[x, z] = (0)$.
Since $A\neq (0)$ and R is prime, the above expression yields that $[x, z] = 0$ for all $x, z\in A$.Now by replacing x with xr, we get $x[r, z] = 0$.Again by replacing x with xs, we get $xs[r, z] = 0$ for all $x, z\in A$ and $r, s\in R$. That is, $xR[r, z] = (0)$.
The primeness of R forces that either $x = 0$ or $[r, z] = 0$, but $A\neq (0)$, we have $[r, z] = 0$.
Now by replacing z with zs to get $z[r, s] = 0$ for all $z\in A$ and $r, s\in R$, this implies that $zR[r, s] = (0)$.
The primeness of R forces that either $z = 0$ or $[r, s] = 0$, but $A\neq (0)$, we have $[r, s] = 0$ for all $r, s\in R$.Hence R is commutative.
Now we assume that $F\neq 0$. For any $x, y\in A$, we have $F(xy) - xy \in U$. This can be rewritten as $F(x)y + xd(y) - xy \in U$.
Now by replacing y by yz, we obtain
$F(x)yz + xd(y)z + xyd(z) - xyz \in U$ for all $x, y, z\in A$.
Thus, in particular
$[(F(x)y + xd(y) - xy)z + xyd(z), z] = 0$, (1)
for all $x, y, z\in A$.
This gives that $[xyd(z), z] = 0$ for all $x, y, z\in A$ and hence
$xy[d(z), z] + x[y, z]d(z) + [x, z]yd(z) = 0$, (2)
for all $x, y, z\in A$.
For any $y_1\in A$, by replacing x by y_1x in 2 and using 2, we get $[y_1, z]xyd(z) = 0$ for all $x, y, z\in A$ and hence $[y_1, z]xRAd(z) = (0)$.
Thus, the primeness of R implies that for each $z\in A$, either $Ad(z) = (0)$ or $[y_1, z]x = 0$. The set $z\in A$ for which these two properties hold are additive subgroups of A whose union is A.Therefore either $Ad(z) = (0)$ for all $z \in A$ or $[y_1, z]x = 0$ for all $x, y_1, z\in A$. If $Ad(z) = (0)$ for all $z \in A$, then $ARd(z) = (0)$ for all $z \in A$. Since $A\neq (0)$ and R is prime, the above expression gives that $d(z) = 0$ for all $z \in A$. This implies that $d(zr) = 0$ for all $z \in A$ and $r\in R$. Hence it follows that $zd(r) = 0$ that is $ARd(r) = (0)$. Since $A\neq (0)$, the primeness of R yields that $d(r) = 0$ for all $r \in R$, a contradiction. On the other hand if $[y_1, z]x = 0$ for all $x, y_1, z\in A$, then $[y_1, z]RA = (0)$ for all $y_1, z \in A$. The primeness of R implies that $[y_1, z] = 0$ for all $y_1, z\in A$ and hence R is commutative.

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Theorem 2 : Let R be a prime nonassociative ring satisfying $[xy, z] = x[y, z] + [x, z]y$ for all x, y, z in R and A be an associative nonzero ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d such that $F(xy) + xy \in U$ for all $x, y \in A$, then R is commutative.

Proof : If F is a generalized derivation satisfying the property $F(xy) + xy \in U$ for all $x, y \in A$, then the generalized derivation $(-F)$ satisfies the *condition* $(-F)(xy) - xy \in U$ for all $x, y \in A$. Using the same aguments as used in Theorem 1, we conclude that R is commutative.

Theorem 3 : Let R be a prime nonassociative ring satisfying $[xy, z] = x[y, z] + [x, z]y$ for all x, y, z in R and A be an associative nonzero ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d such that $F(x)F(y) - xy \in U$ for all $x, y \in A$, then R is commutative.

Proof : By hypothesis $F(x)F(y) - xy \in U$ for all $x, y \in A$.If $F = 0$, then $xy \in U$ for all $x, y \in A$.Using the same arguments as we used in Theorem 1, we conclude that R is commutative.Now we assume that $F \neq 0$.For any $x, y \in A$, we have $F(x)F(y) - xy \in U$.By replacing y with yr, we get

$F(x)(F(y)r + yd(r)) - xyr \in U$. That is,

$(F(x)F(y) - xy)r + F(x)yd(r) \in U$, (3)

for all $x, y \in A$ and $r \in R$.

This implies thatfor all $x, y \in A$ and $r \in R$.

This implies that

$[F(x)yd(r), r] = 0$, (4)

This can be rewritten as

$F(x)[yd(r), r] + [F(x), r]yd(r) = 0$, (5)

for all $x, y \in A$ and $r \in R$.

Substituting $F(x)y$ for y in 5 and using 5, we get

$[F(x),r]F(x)yd(r) = 0$, (6)

for all $x, y \in A$ and $r \in R$. That is,

$[F(x), r]F(x)RAd(r) = 0$.

Thus for each $r \in R$, primeness of R forces that either $[F(x), r]F(x) = 0$ or $Ad(r) = (0)$. The set of $r \in R$ for which these two properties hold form additive subgroups of R whose union is R. Hence either $[F(x), r]F(x) = 0$ for all $x \in A, r \in R$ or $Ad(r) = (0)$ for all $r \in R$. If $Ad(r) = (0)$ for all $r \in R$, then $ARd(r) = (0)$ for all $r \in R$. Since$A \neq (0)$ and R is prime, the above relation yields that $d = 0$ which is a contradiction. Therefore, we assume the remaining possibility that $[F(x), r]F(x) = 0$ for all $x \in A, r \in R$. For any $S \in R$, we replace r by rs. Then $[F(x), r]RF(x) = 0$, for all $x \in A, r \in R$.

The primeness of R implies that either $F(x) = 0$ or $[F(x),r] = 0$.

Thus in both cases we have, $[F(x), r] = 0$, for all $x \in A, r \in R$.

The above relation gives that $F(x) \in U$ for all $x \in A$ and hence $F(x)F(y) \in U$, for all $x, y \in A$. Thus our hypothesis yield that $xy \in U$. Now by using the same arguments as we used in Theorem 1, we can conclude that R is commutative.

Theorem 4 : Let R be a prime nonassociative ring satisfying $[xy, z] = x[y, z] + [x, z]y$ for all x, y, z in R and A be an associative nonzero ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d such that $F(x)F(y) + xy \in U$ for all $x, y \in A$, then R is commutative.

Proof : We can prove this theorem by using the same arguments as in Theorem 3. Hence we conclude that R is commutative.

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